Christ Church Grammar School

2020
TEST 3

MATHEMATICS SPECIALIST Year 12

Section One:
Calculator-free

Your name SOLUTIONS

Teacher's name _____

Time and marks available for this section

Reading time for this section:	2 minutes
Working time for this section:	15 minutes
Marks available:	16 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. The rules of conduct of the CCGS assessments are detailed in the Reporting and Assessment Policy. Sitting this assessment implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet using a blue/black pen. Do not use erasable/gel pens

3. Answer all questions.

4. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

5. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

See next page

Question 1 **(6 marks)**

Determine:

(a) $\int \sin^2 x \, dx$. (2 marks)

$\cos 2x = 1 - 2\sin^2 x$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$ ✓ correctly substitutes trig identity

$= \frac{1}{2}\int (1 - \cos 2x)\, dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2} + c\right]$

$= \frac{1}{2}x - \frac{\sin 2x}{4} + c.$ ✓ correctly integrates

(b) $\int \frac{8}{4-x^2} dx$. (4 marks)

$= \int \frac{8}{(2-x)(2+x)} dx.$

recognises and subst. in partial fraction ✓

$\frac{8}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x}.$

$\Rightarrow 8 = A(2+x) + B(2-x)$

If $x = 2$ $8 = 4A$ $\therefore A = 2$

If $x = -2$ $8 = 4B$ $\therefore B = 2$ ✓ solves for the partial fractions

$= \int \left(\frac{2}{2-x} + \frac{2}{2+x}\right) dx.$

$= -2\ln|2-x| + 2\ln|2+x| + c$ ✓ Integrates correctly

$= 2\ln|2+x| - 2\ln|2-x| + c$

$= 2\ln\left|\frac{2+x}{2-x}\right| + c.$ ✓ Simplifies fully

See next page

Question 2 **(6 marks)**

(a) Solve the system of linear equations below. (3 marks)

$$2x + y - 3z + 7 = 0$$
$$x + z - 7 = 0$$
$$3y - z + 7 = 0$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & -7 \\ 1 & 0 & 1 & 7 \\ 0 & 3 & -1 & -7 \end{array}\right] R_1 - 2R_2$$

$$= \left[\begin{array}{ccc|c} 2 & 1 & -3 & -7 \\ 0 & 1 & -5 & -21 \\ 0 & 3 & -1 & -7 \end{array}\right] R_3 - 3R_2$$

✓ correctly row reduces

$$= \left[\begin{array}{ccc|c} 2 & 1 & -3 & -7 \\ 0 & 1 & -5 & -21 \\ 0 & 0 & 14 & 56 \end{array}\right]$$

$z = 4$

$y = -21 + 5(4)$

$y = -1$

✓ solves for 'z' and 'y'

$x = 7 - 4$

$x = 3$

$\therefore x = 3, y = -1, z = 4.$ ✓ calculates remaining value of x.

See next page

Question 2 continued

(b) Consider another system of linear equations, where one of the coefficients is k, $k \in \mathbb{R}$.

$$x + y + z = 0$$
$$2x + z = 2$$
$$y + kz = -5$$

It can be shown that the solutions, in terms k, to this system of equations are:

$$x = \frac{2k+3}{2k-1} \qquad y = \frac{5-2k}{2k-1} \qquad z = \frac{8}{1-2k}$$

Explain whether this system of equations will have a unique solution for all real values of k. If not, then explain the geometric interpretation of this. (3 marks)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & k & -5 \end{array}\right]$$

$z = \frac{8}{1-2k} \Rightarrow (1-2k)z = 8$

If $k = \frac{1}{2}$ then no solution $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 8 \end{array}\right]$

✓ states no solution when $k = \frac{1}{2}$

So the system will not have an unique solution for all real values of K

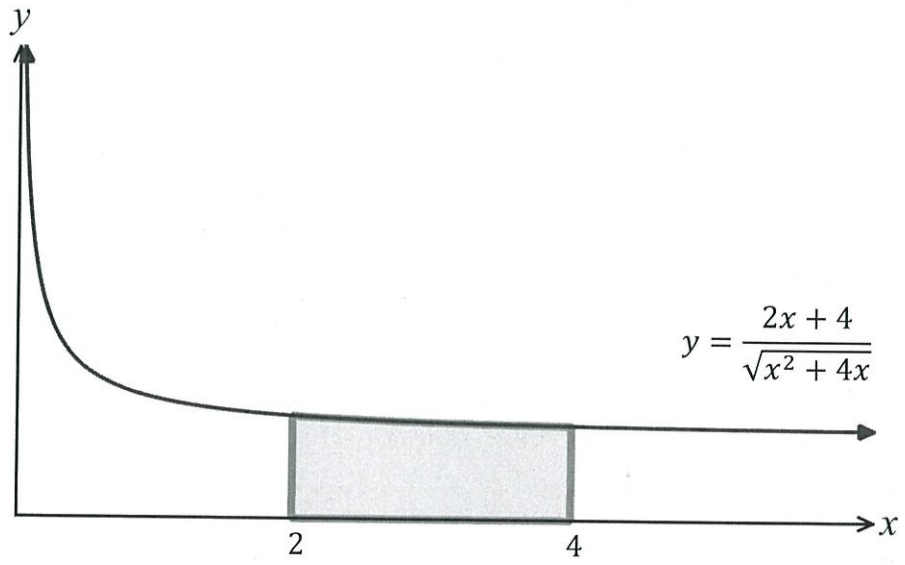
✓ explicitly answers question

If $k = \frac{1}{2}$ then the 3 planes are non-parallel but do not interseet at a single point in space.

✓ geometric interpretation when $k = \frac{1}{2}$ (at no solution)

Question 3 **(4 marks)**

Calculate the exact area of the shaded region shown below in square units.



$\int_2^4 \frac{2x+4}{(x^2+4x)^{1/2}}\,dx$

let $u = x^2 + 4x$

$\frac{du}{dx} = 2x + 4$

$dx = \frac{du}{2x+4}$

$= \int_{12}^{32} \frac{1}{u^{1/2}}\,du$

✓ substs + changes limits or recognises numerator is derivative or denominator

when $x = 2$, $u = 12$

$x = 4$, $u = 32$

$= \int_{12}^{32} u^{-1/2}\,du$

$= \left[\frac{u^{1/2}}{1/2}\right]_{12}^{32}$

$= \left[2\sqrt{u}\right]_{12}^{32}$ ✓ integrates correctly

$= 2\sqrt{32} - 2\sqrt{12}$ ✓ substitutes in limits

$= 8\sqrt{2} - 4\sqrt{3}$ units^2 ✓ simplifies fully

or

$\int_2^4 (2x+4)(x^2+4x)^{-1/2}\,dx$

$= \left[\frac{(x^2+4x)^{1/2}}{1/2}\right]_2^4$

$= 2\sqrt{x^2+4x}\Big]_2^4$

$= 2\sqrt{16+16} - 2\sqrt{4+8}$

$= 2\sqrt{32} - 2\sqrt{12}$

$= 8\sqrt{2} - 4\sqrt{3}$

End of questions

Additional working space

Question number: _____

Additional working space

Question number: _____

Christ Church Grammar School

2020
TEST 3

MATHEMATICS SPECIALIST Year 12

Section Two:
Calculator-assumed

Your name SOLUTIONS

Teacher's name _____

Time and marks available for this section

Reading time for this section:	3 minutes
Working time for this section:	25 minutes
Marks available:	24 marks

Materials required/recommended for this section

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This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
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Special items: drawing instruments, templates, and up to three calculators approved for use in this assessment

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See next page

Question 4 **(7 marks)**

(a) Determine $\int \left((\sec^2(1-x)+\left(\frac{1}{3x-1}\right)\right) dx$. (2 marks)

$= -\tan(1-x) + \frac{1}{3}\ln|3x-1| + c$

✓ correctly calculates $\int$ also accept $\tan(x-1)$

✓ correctly calculates $\int$

(b) Show $\int \frac{x}{2x^2+3x} dx = \frac{1}{2}\ln|2x+3| + c$. (2 marks)

LHS: $\int \frac{\frac{x}{x}}{\frac{2x^2}{x}+\frac{3x}{x}} dx$ ✓ divides numerator & denominator by x.

$= \int \frac{1}{2x+3} dx$ ✓ shows correct simplification

$= \frac{1}{2}\ln|2x+3| + c$

(c) A solid is formed by rotating the curve with equation $y = (x-1)e^{2x}$ between $x = 0$ and $x = 1$ through 2π radians about the x –axis. Determine the exact value of the volume of this solid. (3 marks)

$V = \pi\int_a^b y^2 dx$

$= \pi\int_0^1 ((x-1)e^{2x})^2 dx$ ✓ correct limits ✓ correct subst into formula

$= \frac{\pi}{32}(e^4 - 13)$ units^3 ✓ correct exact value

or $\pi\left(\frac{e^4}{32} - \frac{13}{32}\right)$

or $\frac{\pi e^4}{32} - \frac{13\pi}{32}$

Note: 4.08 (2dp) units^3 Answer only 1 mark if exact

-1 mark if not an exact value.

See next page

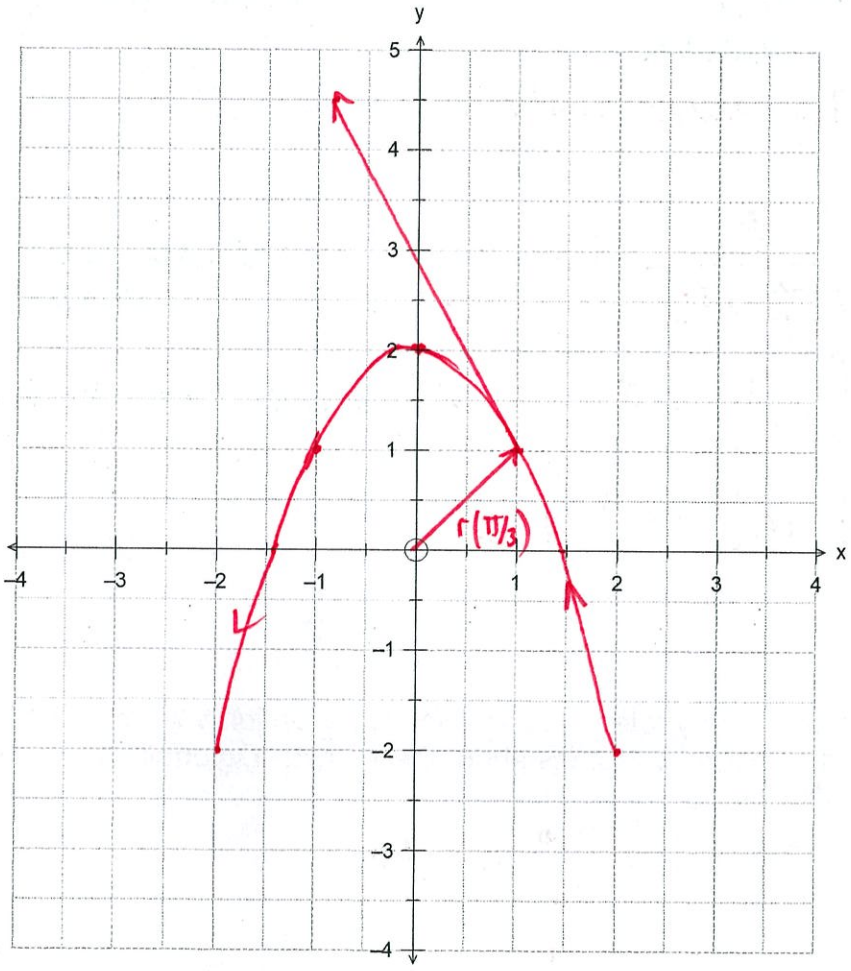
Question 5 **(12 marks)**

The position of a particle in two-dimensional space is given by

$$\boldsymbol{r}(t) = 2\cos t\,\boldsymbol{i} - 2\cos 2t\,\boldsymbol{j},$$

where $|\boldsymbol{r}(t)|$ is given in metres and t in minutes since the motion began.

(a) Draw the path traced by the particle on the axes provided below. (3 marks)



✓ parabola shape

✓ going through (−1,1) and (1,1)

✓ restricts domain to $-2 \le x \le 2$

$x = 2\cos t$

$y = -2\cos 2t$

$\frac{y}{-2} = \cos 2t$

$\frac{x}{2} = \cos t$

$\frac{x^2}{4} = \cos^2 t$

$\therefore \frac{y}{-2} = 2\left(\frac{x^2}{4}\right) - 1$

$y = -x^2 + 2$

$y = 2 - x^2$

(b) State the time it takes for the particle to complete one full cycle of motion. (1 mark)

$t = 2\pi$ ✓

Question 5 continued

(c) Determine the position vector and velocity vector of the particle at the instant where $x = 1$ for the first time. Draw and label these vectors on the same diagram drawn in part (a). (6 marks)

$x = 1 = 2\cos t$

$\frac{1}{2} = \cos t$

$t = \frac{\pi}{3}, \frac{2\pi}{3}$

$\therefore t = \pi/3$ is the first time ✓ calculates correct first time.

$r(\pi/3) = 2\cos(\pi/3)\underset{\sim}{i} - 2\cos(2\pi/3)\underset{\sim}{j}$

$= \underset{\sim}{i} + \underset{\sim}{j}$ ✓ $r(t)$ calculates at $t =$.

$v(t) = -2\sin t\,\underset{\sim}{i} + 4\sin(2t)\,\underset{\sim}{j}$ ✓ determines $v(t)$ correctly

$v(\pi/3) = -2\sin(\pi/3)\underset{\sim}{i} + 4\sin(2\pi/3)\underset{\sim}{j}$

$= -\sqrt{3}\,\underset{\sim}{i} + 2\sqrt{3}\,\underset{\sim}{j}$ ✓ $(-1.7\underset{\sim}{i} + 3.5\underset{\sim}{j})$ determines $v(t)$ at correct t or their t if above incorrect.

✓ one for $r(\pi/3)$ on diagram

✓ one for $v(\pi/3)$ on diagram

(d) Calculate the magnitude of acceleration of the particle at time $t = \frac{\pi}{6}$. (2 marks)

$a(t) = -2\cos t\,\underset{\sim}{i} + 8\cos(2t)\,\underset{\sim}{j}$ ✓ calculates $a(t)$

$a(\pi/6) = -2\cos(\pi/6)\underset{\sim}{i} + 8\cos\left(\frac{2\pi}{6}\right)\underset{\sim}{j}$

$= -\sqrt{3}\,\underset{\sim}{i} + 4\underset{\sim}{j}$ ✓ correct value for $a(\pi/6)$

Answer only 2 marks

$|a(\pi/6)| = \sqrt{(-\sqrt{3})^2 + 4^2}$

$= \sqrt{19}$ m/min.

$= 4.36$ m/min

See next page

Question 6 **(5 marks)**

The position of a particle at time t is given by the parametric equations

$x = t\cos t, \quad y = t\sin t, \quad t \geq 0.$

(a) Determine an expression for the instantaneous speed of the particle. (3 marks)

$r(t) = t\cos t\,\underset{\sim}{i} + t\sin t\,\underset{\sim}{j}$

$v(t) = (\cos t + t(-\sin t))\,\underset{\sim}{i} + (\sin t + t\cos t)\,\underset{\sim}{j}$

$v(t) = (\cos t - t\sin t)\,\underset{\sim}{i} + (\sin t + t\cos t)\,\underset{\sim}{j}$ ✓ determines velocity

$|v(t)| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$ ✓ determines expression for speed

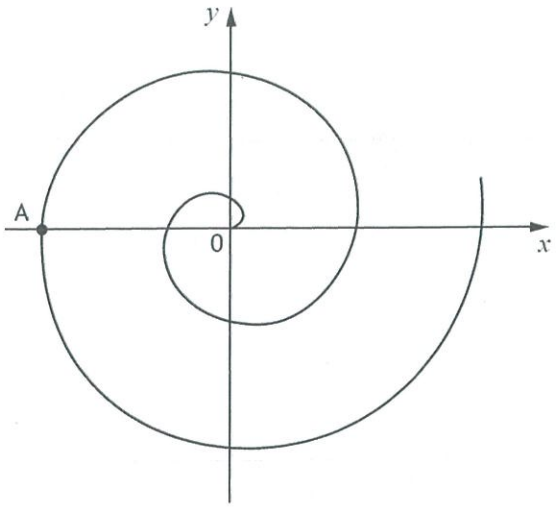
$= \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t}$

$= \sqrt{\cos^2 t + \sin^2 t + t^2(\sin^2 t + \cos^2 t)}$

$= \sqrt{1 + t^2}$ ✓ simplified expression

Question 6 continued

The diagram below shows the path that the particle takes.



(b) Calculate the instantaneous speed of the particle at point A. (2 marks)

at point A $y = 0$

$0 = t\sin t$ $t = 0$ or $\sin t = 0$

$t = 0, \pi, 2\pi, 3\pi, 4\pi$

at point A $t = 3\pi$

$|v(3\pi)| = \sqrt{1 + (3\pi)^2}$

$= \sqrt{1+9\pi^2}$

or

9.47768

$\simeq 9.5$ units/time

✓ determines correct t value for point A

✓ determines speed at $t = 3\pi$. (accept exact or rounded (any dp))

Answer only 2 marks.

End of questions

Additional working space

Question number: _____

Additional working space

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